Question 1 (*8 points*). Find *all solutions* of the following system of equations.

$$\begin{aligned} x+y &= z+w+1 \\ x+z &= y+w+2 \\ x+w &= y+z+3 \end{aligned}$$

Question 2 (*8 points*). For which values of x is the matrix

$$A = \begin{bmatrix} 1 & -2 & -3 \\ 2 & x-1 & 4 \\ x+2 & -1 & 2 \end{bmatrix}$$

non-invertible (singular)? Justify your answer!

Question 3 (*10 points*). Let

$$S = \left\{ \begin{bmatrix} -s \\ s - 5t \\ 3t + 2s \end{bmatrix} : s, t \in \mathbb{R} \right\}.$$

1. Show that S is a *subspace* of $\mathbb{R}^3$.
2. Find a *basis* for S.
3. What is the *dimension* of S?
4. Find a *subspace* D of S such that $\dim(D) = 1$.

Question 4 (*8 points*). Suppose that $\{u, v\}$ is a linearly independent set in a vector space V. Show that the set $\{u - v, u + v\}$ is *linearly independent*.

Question 5 (*8 points*). Let $T : \mathcal{P}_2 \longrightarrow M_{2\times 2}$ be the linear transformation given by

$$T(ax^2 + bx + c) = \begin{bmatrix} c & a - 2b \\ 2b - a & c \end{bmatrix}$$

(a). Find a *basis* for the *null space* $N(T)$.
(b). Find a *basis* for the *range space* $R(T)$.

Question 6 (*8 points*). Let T and S be linear transformations such that

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 2x + y + z \\ 3y + 3z \\ 4x \end{bmatrix} \text{ and } S\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 2x + 2z \\ 2y - 2z \\ z \end{bmatrix}$$

Which of T and S are *isomorphisms*? For the one(s) that is/are find the *matrix representation* of the *inverse transformation*.

Question 7 (*8 points*). (a) Find the *characteristic polynomial* and the *eigenvalues* of the matrix

$$A = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

(b). Find the *eigenvalue* corresponding to the eigenvector $\begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}$ of the matrix $\begin{bmatrix} 1 & 1 & 1 & 0 \\ -1 & -1 & 0 & -1 \\ -1 & 1 & 0 & 1 \\ 0 & -1 & -1 & 0 \end{bmatrix}$

Question 8 (*8 points*). Consider the matrix

$$A = \begin{bmatrix} 1 & 2 & -2 \\ -2 & 5 & -2 \\ -6 & 6 & -3 \end{bmatrix}$$

(a). *Diagonalize* A, if possible. That is, if possible, find an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$.

(b). Use part (a) to find A^{10}. [Do not simplify]

Question 9 (*10 points*). The set $B = \left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} \right\}$ is a basis for $\mathbb{R}^3$.

(a). Use the *Gram-Schmidt process* to find an *orthonormal basis* N from B.

(b). Express $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ as a *linear combination* of the vectors of N.

Question 10 (*24 points*). Identify the most likely solution.

1. If A and B are $n \times n$ matrices, and $\det(A) = 3$, $\det(B) = 5$, then $\det(2A + B)$ is
   (a) 8
   (b) 11
   (c) 17
   (d) 30
   (e) impossible to determine from the information given.

2. The homogeneous matrix equation $A\mathbf{x} = \mathbf{0}$
   (a) is consistent and may have infinitely many solutions.
   (b) is consistent and always has a unique solution.
   (c) is either inconsistent or has a unique solution.
   (d) is either inconsistent or has infinitely many solutions .
   (e) must be inconsistent

3. The angle between vectors $\mathbf{u} = (1, -1, 0)$ and $\mathbf{v} = (-1, 0, 1)$ is
   (a) $\pi/4$
   (b) $\pi/3$
   (c) $\pi/2$
   (d) $2\pi/3$
   (e) π

4. The set $\{1, 1 + x, 1 + x^3\} \subseteq \mathcal{P}_3$
   (a) is linearly dependent but does not span $\mathcal{P}_3$.
   (b) is linearly dependent and spans $\mathcal{P}_3$.
   (c) is linearly independent but does not span $\mathcal{P}_3$.
   (d) is linearly independent and spans $\mathcal{P}_3$.
   (e) may be any of the above.

5. If $T : \mathbb{R}^8 \to \mathbb{R}^3$ is linear and onto, then the dimension of the nullspace of T is
   (a) 3
   (b) 4
   (c) 5
   (d) 8
   (e) impossible to determine from the information given.

6. If $T : \mathbb{R}^3 \to \mathbb{R}^2$ is linear, and $T\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $T\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

then $T\left(\begin{bmatrix} 12 \\ 22 \\ 32 \end{bmatrix}\right) =$

(a) $\begin{bmatrix} 24 \\ 6 \end{bmatrix}$

(b) $\begin{bmatrix} 25 \\ 7 \end{bmatrix}$

(c) $\begin{bmatrix} 26 \\ -8 \end{bmatrix}$

(d) $\begin{bmatrix} 28 \\ -10 \end{bmatrix}$

(e) $\begin{bmatrix} 30 \\ 12 \end{bmatrix}$

7. The inverse of matrix $A = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}$ is

(a) $\begin{bmatrix} 4 & -3 \\ -2 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & -3 \\ -2 & 4 \end{bmatrix}$

(c) $\begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$

(d) $\begin{bmatrix} -1/2 & 3/2 \\ 1 & -1/2 \end{bmatrix}$

(e) undefined (A is singular).

8. If A is a diagonalizable 8×8 matrix with characteristic polynomial $(\lambda - 1)^4(\lambda - 4)^3(\lambda - 7)$, Then Rank$(A - 4I)$ is

(a) 2

(b) 3

(c) 5

(d) 6

(e) 8

9. If A is a 2×3 matrix and AB is a 2×5 matrix, then B is

(a) 2×5
(b) 5×2
(c) 3×5
(d) 5×3
(e) none of the above

10. The system of equations with augmented matrix

$$\begin{bmatrix} 4 & -3 & 8 & 2 & 11 \\ 0 & -2 & 1 & 7 & b \\ 0 & 0 & 0 & b^2-4 & b-2 \end{bmatrix}$$

is consistent for

(a) all values of b .
(b) all values of b except 0 .
(c) all values of b except 2.
(d) all values of b except 2 and -2.
(e) all values of b except 0, 2 and -2.

11. The rank of the matrix

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 6 & 7 & 8 & 9 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}$$

is

(a) 1
(b) 2
(c) 3
(d) 4
(e) 5

12. Let A and B be invertible $n \times n$ matrices, then $(AB^T)^{-1}A(AB)^T =$

(a) A
(b) A^T
(c) AB^T
(d) $A^{-1}B^TA^T$
(e) $A^{-1}(B^T)^{-1}AA^TB^T$
(f) $(B^T)^{-1}A^TB^T$